Tsemo Aristide
103-135 Leeward Glenway
North York
M3C 2Z6
tsemoaristide@hotmail.com

KAHLER, POISSON GEOMETRY OF CR LIE GROUPS.

"Las frutas de la honestidad se recogen
en muy poco tempo y duran para siempre."
Colombian Proverb.

Abstract.

A Cauchy Riemann (CR) Lie group is a Lie group G which Lie algebra $\mathcal{G}$ has a vector subspace $\mathcal{H}$ endowed with an endomorphism j such that $j^2 = -Id$, and for each elements x, y in $\mathcal{H}$, we have $[j(x), j(y)] - [x, y]$ is an element of $\mathcal{H}$, and $[j(x), j(y)] - [x, y] = j([j(x), y] + [x, j(y)])$. In this paper we study the geometry of a CR Lie groups G when its Lie algebra $\mathcal{G}$ is endowed with more geometric structures compatible with j, as kahler, and poisson type structures.

0. Introduction.

Let (M, j) be a complex manifold, and H an hypersurface of M, for each element x of H the tangent space TH_x of H at x is endowed with a maximal complex vector space $E_x = TH_x \cap j(TH_x)$. The collection of vector spaces E_x defines a vector bundle E such that for sections X, and Y of E, we have:

$$[j(X), j(Y)] - [X, Y] \in E$$

since (M, j) is a complex structure, we also have:

$$[j(X), j(Y)] - [X, Y] = j([j(X), Y] + [X, j(Y)]).$$

More generally, a Cauchy Riemann (CR) structure on a manifold M, is defined by a subbundle H of TM endowed with an endomorphism j such that:

$$j^2 = -Id_H$$

For each sections X and Y of H, we have:

$$[j(X), j(Y)] - [X, Y] \in H$$

$$[j(X), j(Y)] - [X, Y] = j([j(X), Y] + [X, j(Y)],$$

The notion of CR manifolds is studied by many authors see[3]

In this paper we study Left-invariant $CR-$structures on Lie groups compatible with geometric properties as poisson, and kahler type properties. More precisely:

Definition 0.1.

A kahlerian complex real Lie group $(G, \mathcal{H}, j, <,>)$, is a Lie group G endowed with a CR structure defined by the vector subspace $\mathcal{H}$, j is an endomorphism of $\mathcal{G}$ which image is $\mathcal{H}$. We suppose that the following properties are verified:

1. j preserves $\mathcal{H}$, the restriction of j^2 to $\mathcal{H}$ is $-Id$.

2.$[X,Y] - [jX, jY] \in \mathcal{H}$ if $X, Y \in \mathcal{H}$

3.$[j(X), j(Y)] = [X,Y] + j([X, j(Y)] + [j(X), Y]) if X, Y \in \mathcal{H}$

Moreover we will suppose that there exists a left-invariant riemannian metric on G, defined by a scalar product on $\mathcal{G}$ such that $\omega =<, j>$ is closed. This means that ω is antisymmetric and

$$\omega([x,y],z) + \omega([z,x],y) + \omega([y,z],x) = 0$$

We remark that the restriction of ω to $\mathcal{H}$ is not degenerated.

We show results related to those structures, for example, we show that a semi-simple kahlerian CR Lie group such that the codimension of $\mathcal{H}$ is 1 is locally isomorphic to $so(3)$ or $sl(2)$.

1. The structure of kahlerian Lie groups.

Let G be Lie group. A left symmetric structure on G, is defined by a product on its Lie algebra $(\mathcal{G}, [,])$

$$\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$$

$$(x,y) \longrightarrow xy$$

which verified

$$xy - yx = [x,y]$$

and

$$x(yz) - (xy)z = y(xz) - (yx)z.$$

This is equivalent to endows G with a left invariant connection which curvature and torsion forms vanish identically.

In this part we consider a CR kahlerian Lie group $(G, \mathcal{H}, j, <,>)$. The Lie algebra $\mathcal{G}$ of G, will be call a CR kahlerian Lie algebra.

We will define the following product on $\mathcal{H}$:

For x, y, $z \in \mathcal{H}$, we set

$$\omega(xy, z) = -\omega(y, [x,z]) = \omega([x,z], y) =< [x,z], j(y) >$$

Proposition 1.1.

For every x, y, u and z in $\mathcal{H}$ we have

$$\omega(xy - yx, u) = \omega([x, y], u) \tag{1}$$

and if the bracket $[x, y]' = xy - yx$ satisfies the jacobi identity

$$x(yz) - (xy)z = y(xz) - (yx)z \tag{2}$$

Proof.

The proof almost copy the one of left-invariant symplectic structures on Lie groups.

Let u be an element of $\mathcal{H}$, for x, y, z in $\mathcal{H}$, we have $\omega(xy - yx, u) = -\omega(y, [x, u]) + \omega(x, [y, u])$ then (1) follows from the definition of ω (the closed property).

Now we prove the second assertion:

$$\omega(x(yz), u) = -\omega(yz, [x, u]) = \omega(z, [y, [x, u]']')$$

We also have:

$$\omega((xy)z, u) = -\omega(z, [xy, u]) = \omega(z, [u, xy]) = -\omega(uz, xy)$$

This implies that

$$\omega(x(yz) - (xy)z, u) - \omega(y(xz) - (yx)z, u) =$$

$$\omega(z, [y, [x, u]] - [x, [y, u]]) + \omega(uz, xy - yx)$$

The property (1) implies that:

$$\omega(uz, xy - yx) = \omega(uz, [x, y]') = -\omega(z, [u, [x, y]']')$$

We deduce that

$$\omega(x(yz)-(xy)z, u)-\omega(y(xz)-(yx)z, u) = \omega(z, [y, [x, u]']'+[x, [u, y]']'+[u, [y, x]']') = 0$$

Corollary 1.2.

Let $\mathcal{G}, \mathcal{H}, j)$ be a kahlerian $cr-$algebra, then the product $(x, y) \to xy - yx = [x, y]'$ defined on $\mathcal{H}$ a structure of a kahlerian Lie algebra if it satisfies the Jacobi identity.

Proof.

We deduce from the property (2) that the product defined on $\mathcal{H}$, $(x, y) \to xy$ endows $\mathcal{H}$ with a structure of Left symmetric algebra which underlying Lie algebra is $[,]'$, the morphism j is defines also a complex structure on H, and the scalar product $<, >$ a kahlerian structure.

Proposition 1.3.

Consider the vector subspace $\mathcal{L}$ such that for each $x \in \mathcal{L}$, we have $\omega(x, \mathcal{G}) = 0$, $\mathcal{L}$ is a Lie subalgebra of $\mathcal{G}$ and is the $<,>$orthogonal vector space of $\mathcal{H}$.

Proof.

Let x and y two elements of $\mathcal{L}$, for every element z of $\mathcal{G}$ we have:

$$\omega([x, y], z) = \omega(y, [z, x]) + \omega(x, [y, z])$$

since x and y are elements of $\mathcal{L}$, we deduce that $\omega([x, y], z) = 0$, and that $[x, y] \in \mathcal{L}$.

For $x \in \mathcal{L}$, and $y \in \mathcal{H}$, we have:

$$\omega(x, y) =< x, j(y) >= 0,$$

Since the restriction of j to $\mathcal{H}$ is an automorphism, we deduce that $\mathcal{L}$ and $\mathcal{H}$ are $<,>$ orthogonal each other

Proposition 1.4.

Let L be the Lie subgroup which Lie algebra is $\mathcal{L}$, and M the right quotient G/L, then M is a kahlerian manifold.

Proof.

The fact that $\mathcal{H}$ and $\mathcal{L}$ are orthogonal each other implies that the riemannian metric $<,>$ gives rise to a metric $<,>'$ of M, the morphism j also gives rise to a complex structure j' of M. Denote by p the projection $p : G \rightarrow G/L$, we have $\omega = p^* <, j' >'$. This implies that $\omega' =<, j' >'$ is a symplectic form defined on M, thus (M, ω', j') is a kahlerian manifold.

Let $(\mathcal{H}, [,]', j', <,>')$ be a kahlerian Lie algebra, and V a vector space. Supposed defined a Lie algebra structure on $\mathcal{G} = \mathcal{H} + V$ such that there exists a map:

$$\alpha : \mathcal{H} \times \mathcal{H} \rightarrow V$$

such that for $x, y \in \mathcal{H}$ we have $[x, y] = [x, y]' + \alpha(x, y)$, suppose that $\alpha(j(x), j(y)) = \alpha(x, y)$.

We suppose also that there exists a scalar product $<,>$ on $\mathcal{G}$ which extends $<,>'$ such that $\mathcal{H}$ and V are orthogonal, we also extend j' to an endomorphism j of $\mathcal{G}$ such that $j(V) = 0$. We suppose that the form $\omega =<, j >$ is closed, then $(\mathcal{G}, j, <,>)$ is a $CR-$kahlerian algebra.

Remark that the fact that $\mathcal{G}$ is a kahlerian $CR-$Lie algebra implies the following property:

$$\oint \alpha([x, y]', z) + [\alpha(x, y), z] = 0$$

Remark that if $\mathcal{H}$ is a sub Lie algebra of $\mathcal{G}$, then its symplectic structure defined on $\mathcal{G}$ a left invariant Poisson structure.

Examples of kahlerian $CR-$structures.

1.

Consider the $n-$dimensional commutative Lie algebra $I\!R^n$ endowed with its flat riemannian metric $<,>$, and V an even dimension subspace of $I\!R^n$ endowed with a linear map j such that $j^2 = -id$, then $(I\!R^n, V, j, <,>)$ is a Lie kahlerian $CR-$algebra.

2.

Consider the semi-simple algebra $so(3)$, and (e_1, e_2, e_3) its basis in which its Lie structure is defined by

$$[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1$$

On $Vect(e_1, e_2)$ the subspace generated by e_1 and e_2, we consider the linear map j defines by $j(e_1) = e_2$ and $j(e_2) = -e_1$.

Let $<,>$ be the scalar product defined on $so(3)$ by $< e_i, e_j >= \delta_{ij}$,

The family $(so(3), V, j, <,>)$ is a $cr-$kahlerian Lie algebra.

Proposition 1.6.

Let $Z(\mathcal{G})$ be the center of $\mathcal{G}$, then $U = (Z(\mathcal{G}) \cap \mathcal{H}) + j(Z(\mathcal{G}) \cap \mathcal{H})$ is a Lie commutative algebra and for every element z of U, $adz(\mathcal{H}) \subset \mathcal{H}$.

Proof.

Let z and z' be elements of $Z(\mathcal{G}) \cap \mathcal{H}$, we have $[j(z), j(z')] = [z, z'] + j([z, j(z')] + [j(z), z'])$. Since z and z' are in the center of $\mathcal{G}$, we deduce that $[j(z), j(z')] = 0$.

The fact that $[z, \mathcal{H}] \subset \mathcal{H}$ follows from the fact that for x, y in $\mathcal{H}$, we have $[x, j(y)] + [j(x), y]$ is an element of $\mathcal{H}$.

Proposition 1.7.

Let $\mathcal{G}$ be a $CR-$algebra, suppose that there exists an ideal I supplementary to $\mathcal{H}$, then $\mathcal{H}$ is endowed with a complex Lie structure.

Proof.

The projection $p : \mathcal{G} \to \mathcal{H}$ parallel to I defines on $\mathcal{H}$ a Lie-complex structure.

Conversely suppose given an extension

$$0 \longrightarrow I \longrightarrow \mathcal{G} \longrightarrow \mathcal{U} \longrightarrow 0$$

where $\mathcal{U}$ is a Lie algebra endowed with a complex structure, then a supplementary space $\mathcal{H}$ of I defines a $CR-$structure on $\mathcal{G}$ if and only if for every x, y in $\mathcal{H}$, $[jx, jy] - [x, y]$ is an element of $\mathcal{H}$ and $[j(x), j(y)] - [x, y] = j([j(x), y] + [x, j(y)])$, where j is the pulls-back of the complex structure of $\mathcal{U}$.

Proposition 1.8.

Let $(G, \mathcal{H} <,>, j)$ be a CR kahlerian Lie group. Suppose that $\mathcal{H}^\perp = kerj$ is an ideal, then $\mathcal{G}$ is not semi-simple.

Proof.

We have seen that if $\mathcal{H}^\perp$ then $\mathcal{H}$ is endowed with a structure of a kahlerian Lie algebra which is known not be semi-simple. Since the quotient of $\mathcal{G}$ by $\mathcal{H}^\perp$ is isomorphic as a Lie algebra to $\mathcal{H}$, we deduce that $\mathcal{G}$ is not semi-simple.

Proposition 1.9.

Suppose that $\mathcal{H}^{\perp}$ is an ideal, then G is the product of a family of groups G_i, where G_0 is flat, and for $i \geq 1$, the holonomy of the riemannian structure of G_i is irreducible. Each group G_i is endowed with a kahlerian $CR-$structure defined by the subspace $\mathcal{H}_i$ of the Lie algebra $\mathcal{G}_i$ of G_i such that the sum of the dimension of $\mathcal{H}_i$ is $\mathcal{H}$.

Proof.

Suppose that $\mathcal{H}^{\perp}$ is an ideal, then the quotient $\mathcal{L}$ of $\mathcal{G}$ by $\mathcal{H}^{\perp}$ is a kahlerian Lie algebra. The theorem (Lichnerowicz Medina [12]), implies that $\mathcal{L} = \sum \mathcal{L}_i$ where each $\mathcal{L}_i$ is a kahlerian Lie algebra. Now consider the De Rham decomposition of G as a product of groups G_i. The projection $G \rightarrow L$ respect this decomposition. Suppose that the projection $p : \mathcal{G}_i \rightarrow \mathcal{L}_i$ is not trivial, then the orthogonal of the kernel of p defines $\mathcal{H}_i$.

Consider the set of functions $C^{\infty}(G_H)$ defined on G such that $d^n f \in S^n(T^*\mathcal{H})$ where $T\mathcal{H}$ is the left invariant distribution on G defined by $\mathcal{H}$. For each $f \in C^{\infty}(G_H)$, there exists a vector field X_f such that

$$\omega(X_f, .) = df$$

we will denote by $\{f, g\} = \omega(X_f, X_g)$.

Proposition 1.10.

The algebra $(C^{\infty}(G_H), \{,\}$ is a Poisson algebra, i.e $\{,\}$ verifies the Jacobi identity.

2. Homogeneous Kahler CR manifold.

Let $(G, \mathcal{H}, j, <,>)$ be a kahler CR Lie group and Γ a cocompact discrete subgroup of G, the manifold $M = G/\Gamma$ inherits a CR structure from G.

The orbits of the left action of the group L on G defines a foliation $\mathcal{F}_L$ on M. (The Lie algebra of L is $\mathcal{H}$ $<,>$ orthogonal.

Proposition 2.1.

The orbits of the foliation $\mathcal{F}_L$ are closed if and only if the group ΓL is closed in G, in this case M is the total space of a fibration over a kahler manifold.

Proof.

Suppose that the group ΓL is closed, then the quotient $G/\Gamma L$ is a Kahlerian manifold N, the projection map $M \rightarrow N$ induced by the identity map of G which is the given fibration.

Conversely suppose that the orbits of the foliation $\mathcal{F}_L$ are closed. Consider a sequence g_n of ΓL which converges towards the element g of G. Consider a neighbourhood U of g such that the restriction of the projection $p : G \rightarrow M$ to U is injective, then $p(g)$ is an element of the adherence of $p((\Gamma L)e) = \mathcal{F}_{p(e)}$, where e is the neutral element of G. Since we have supposed that the leaves of $\mathcal{F}_L$ are closed, $p(g)$ is an element of $\mathcal{F}_{p(e)}$ which means that ΓL is closed in G. We can thus define the Kahler manifold $G/\Gamma L$.

Deformation of CR kahlerian structures of homogeneous manifolds.

Let $(G, \mathcal{H}, j, <,>)$ be a Lie group endowed with a CR kahlerian structure. Consider a cocompact subgroup Γ of G the manifold $M = G/\Gamma$ inherits from G a CR structure. In this section, we will define the deformation of those structures from two points of view.

Supposed fixed the CR kahlerian structure of G, and a compact manifold M, Let Γ be a group we consider the set of representations $R(\Gamma, G)$ such that for each $u \in R(\Gamma, G)$, u is injective and $G/u(\Gamma)$ is a compact manifold.

To elements u and u' of $R(\Gamma, G)$ will be said equivalent if and only there exists an element g of G such that $u' = gug^{-1}$. We denote by $Def_1(\Gamma, G, \mathcal{H}, j, <,>)$ the space of equivalence classes of those CR kahlerian structures.

Now consider $RCK(G)$, the set of real complex kahlerian structures of G, then for a cocompact subgroup Γ of G $M = G/\Gamma$ inherits a CR kahlerian structure for each element u of $RCK(G)$ denotes by (M, u). We will say that (M, u_1) is equivalent to (M, u_2) if and only if there exists an isomorphism of CR kahlerian complex manifolds between (M, u_1) and (M, u_2). and denote by $Def_2(M, G)$ the set of those CR structures.

3. Kahlerian codimension 1 $CR-$structures.

Theorem 3.1.
Suppose that the codimension of $\mathcal{H}$ is l, and G is semi-simple then G is a Lie group of rank $\leq l$.

Proof.
Let $(\mathcal{G}, \mathcal{H}, j, <,>)$ be a Lie semi-simple algebra endowed with a codimension l $CR-$kahlerian structure. This means that the codimension of $\mathcal{H}$ in $\mathcal{G}$ is l.

The map $\mathcal{G} \to \mathcal{G}^*$,

$$X \longrightarrow \omega(X, .)$$

is a $1-$cocycle for the coadjoint representation. Since $\mathcal{G}$ is semi-simple, this cocycle is trivial. There exists an element α of $\mathcal{G}^*$ such that for $x, y \in \mathcal{G}$ we have:

$$\omega(x, y) = \alpha([x, y])$$

Let K be the Killing form of $\mathcal{G}$, there exists $X \in \mathcal{G}$ such that for each $Y \in \mathcal{G}$ we have:

$$K(X, Y) = \alpha(Y)$$

The Lie algebra $\mathcal{L} = \{x, : \omega(x, \mathcal{G}) = 0\}$ is a dimension l subalgebra of $\mathcal{G}$ since the codimension of $\mathcal{H}$ is l. The Lie algebra $\mathcal{L}$ is the Lie algebra of the subgroup L of G which preserves α, L is also the subgroup which preserves X since K is invariant by the adjoint representation. We deduce that the rank of G is less or equal than l, and then that G is isomorphic to $sl(2)$ or $so(3)$ if the codimension of $\mathcal{H}$ is 1.

Corollary 3.2.

Suppose that $\mathcal{G}, \mathcal{H}, <,>, j)$ *is a semi-simple codimension* 1 *cr*$-$*structure (the codimension of* $\mathcal{H}$ *is* 1*), then* $\mathcal{G}$ *is* $so(3)$ *or* $sl(2)$*.*

4. Poisson $CR-$structures.

Let M be a manifold, and TH and TU two supplementary subbundles of its tangent bundle TM.

Definition 4.1.

An $(TH, TU)-$pseudo-Poisson structure on M is defined by a bivector $\Lambda \in \Lambda^2 TM$ such that

$$[\Lambda, \Lambda] \in TU\Lambda^2 TM$$

where $[\Lambda, \Lambda]$ is the schouten product of Λ by Λ. The bivector Λ defined on $C^\infty(M)$ the bracket $\{,\}$ by the formula

$$\{f, g\} = \Lambda(df, dg)$$

A morphism $f : (M, \Lambda) \rightarrow (M', \Lambda')$ is a differentiable map which commutes with $\{,\}$.

Suppose that the distribution TH defines on M a *cr*$-$structure, we will say that (M, TH, TU, j) defines a pseudo-Poisson *cr*$-$structure on M, if j preserves Λ.

Let G be a Lie group which Lie algebra is $\mathcal{G}$. Consider a subspace H of $\mathcal{G}$ and U a supplementary space to H. The vector spaces H and U define on G right invariant distributions TH and TU.

Definition 4.2.

A pseudo-Lie Poisson structure on G is a bivector $\Lambda \in \Lambda^2 TG$ such that

$$[\Lambda, \Lambda] \in TU\Lambda^2 TG.$$

Moreover we suppose that Λ is multiplicative i.e that the product $G \times G \rightarrow G$ is a morphism of pseudo-Poisson structures.

The bracket $\{,\}$ defined by Λ satisfies the following properties:

$$\{f, f'\}(xy) = \{f \circ L_x, f' \circ L_x\}(y) + \{f \circ R_y, f' \circ R_y\}(x).$$

If we denote by $T_u L_x$ the differential of L_x in u, we have

$$\Lambda(xy) = T_y L_x \Lambda_y + T_x R_y \Lambda_x$$

Consider the tensors $\Lambda_R(x) = T_x R_{x_{-1}} \Lambda_x$ and $\Lambda_L(x) = T_x L_{x_{-1}} \Lambda_x$

Proposition 4.3.

The fact that π *is multiplicative is equivalent to*

1. π_R *is a* $1-$*cocycle for the adjoint representation* $G \rightarrow \Lambda^2 \mathcal{G}$*,i.e* $\pi_R(xy) = \pi_R(x) + Ad_x(\pi_R(y))$.

2. π_L is a $1-$cocycle for the adjoint action of the opposite group of G in $\Lambda^2 G$ i.e $\pi_L(xy) = \pi_L(y) + Ad_{y^{-1}}(\pi_L(x))$.

Let r be an element of $\Lambda^2\mathcal{G}$, r_-, and r_+ the left and right invariant tensors defined by r. We denote by π the tensor $r_+ - r_-$.

The tensor π defines a pseudo-Poisson structure if and only if $[\pi,\pi] = [r,r]_+ - [r,r]_- \in TU\Lambda^2 TG$, or equivalently if

$$Ad_x[r,r] - [r,r] \in U\Lambda^2\mathcal{G}.$$

Suppose that H defines on G a $cr-$structure. We will say that (G, H, Λ, j) is a $cr-$Poisson structure if j preserves Λ.

Moreover we assume that Λ is invariant by j.

Remark.

Suppose defined the $cr-$Poisson structure $(\mathcal{G}_1, \mathcal{H}_1, j_1, \Lambda_1)$ and $(\mathcal{G}_2, \mathcal{H}_2, j_2, \Lambda_2)$, then the tensor $\Lambda_1 \times \Lambda_2$ defines a $cr-$structure $(\mathcal{G}_1 \times \mathcal{G}_2, \mathcal{H}_1 \times \mathcal{H}_2, \Lambda \times \Lambda_2, j_1 \times j_2$ called the product of the Poisson $cr-$structures.

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